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# A Note on $L_1$-Approximations by Exponential Polynomials and Laguerre Exponential Polynomials*

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DEFINITIONS: In this paper, an *exponential polynomial* on $R_+$ (the nonnegative half-line) is a function on $R_+$ of the form

$$\sum_1^n a_k \exp(-kx),$$

and a *Laguerre exponential polynomial* on $R_+$ is a function on $R_+$ of the form

$$p(x) \exp(-x),$$

where $p$ is a polynomial (in the usual sense).

It is the purpose of this note to present a constructive approach to the $L_1$-approximation of integrable functions on $R_+$ by exponential polynomials and by Laguerre exponential polynomials. Our study was motivated by consideration of certain systems of Wiener–Hopf integral equations with a matrix kernel, the entries of which are integrable functions on $R$, the real line [1]. The desirability of approximating integrable functions by exponential polynomials and by Laguerre exponential polynomials was pointed out by Gohberg and Krein [1], who observed that the "standard" factorization of a matrix, the elements of which are rational functions, can be achieved by known algebraic methods. Although consideration of Wiener–Hopf integral equations was our original motivation, the problem of $L_1$-approximation by exponential polynomials and by Laguerre exponential polynomials is of independent interest.

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The problem is as follows: Given a function $g$ in $L_1(R_+)$, find a sequence of exponential polynomials and a sequence of Laguerre exponential polynomials converging to $g$ in the $L_1$-sense. Since we can easily construct a sequence of measurable bounded functions with compact supports converging to $g$ in $L_1$, we can (and shall) assume that $g$ is bounded and has compact support.

LEMMA 1. (i) *The set of functions of the form*

$$\sum_{k=1}^{N} a_k \exp(-2kx), \qquad N = 1, 2,...,$$

*is dense in* $L_2(R_+)$.

(ii) *The set of functions of the form*

$$p(x) \exp(-2x), \qquad p(x) \text{ a polynomial,}$$

*is dense in* $L_2(R_+)$.

*Proof.* (i) Let $u$ be a continuous function on $R_+$ with a compact support. Let $v(x) \equiv u(x) \exp(2x)$. Then $v$ is continuous with a compact support. By the Stone–Weierstrass theorem, there exists a sequence $(w_n)$ of functions of the form

$$\sum a_k \exp(-4kx) \qquad \text{(finite sum)}$$

which converges uniformly on $R_+$ to $v$. It follows that

$$\int_0^\infty | u(x) - w_n(x) \exp(-2x) |^2 \, dx = \int_0^\infty | v(x) - w_n(x) |^2 \exp(-4x) \, dx \to 0.$$

This proves (i), since the continuous functions with compact supports are dense in $L_2(R_+)$.

(ii) Let $u \in L_2(R_+)$ be such that

$$\int_0^\infty u(x) \, p(x) \exp(-2x) \, dx = 0 \tag{2}$$

for every polynomial $p$. Define

$$F(z) = \int_0^\infty u(x) \exp(-2zx) \, dx, \qquad \text{Re } z > 0. \tag{3}$$

Then, $F$ is analytic in the open right half-plane. By (2),

$$F^{(n)}(1) = 0, \qquad n = 0, 1, 2,...,$$

where $F^{(n)}$ is the $n$-th (complex) derivative of $F$, $n \geqslant 1$, and $F^{(0)}$ is $F$. Hence,

$$F(z) = 0 \qquad \text{for} \quad \operatorname{Re} z > 0. \tag{4}$$

In particular,

$$F(k) = 0, \qquad k = 1, 2, \ldots .$$

By part (i), $u = 0$ almost everywhere. This proves (ii). Thus Lemma 1 is proved.

THEOREM 1. *Let $(p_n)$ be a sequence of functions of the form*

$$p_n(x) \equiv \sum_{k=1}^{n} a_k \exp(-2kx),$$

*or of the form*

$$p_n(x) \equiv g_n(x) \exp(-2x) \qquad (g_n(x) \text{ a polynomial}),$$

*converging in $L_2(R_+)$ to $h(x) = g(x) \exp(x)$, where $g$ is a measurable bounded function on $R_+$ with a compact support. Then the sequence $(g_n)$, where $g_n(x) = p_n(x) \exp(-x)$, converges in $L_1(R_+)$ to $g$.*

*Proof.* We have, by Schwarz's inequality,

$$\int_0^\infty | h(x) - p_n(x) | e^{-x} \, dx \leqslant \left( \int_0^\infty | h(x) - p_n(x) |^2 \, dx \right)^{1/2} \left( \int_0^\infty e^{-2x} \, dx \right)^{1/2}, \tag{5}$$

i.e.,

$$\int_0^\infty | g(x) - g_n(x) | \, dx \leqslant \left( \int_0^\infty | h(x) - p_n(x) |^2 \, dx \right)^{1/2} \left( \int_0^\infty e^{-2x} \, dx \right)^{1/2}. \tag{6}$$

Since the right-hand side of (6) tends to 0, the theorem is proved.

Theorem 1 allows us to construct a sequence of exponential polynomials and a sequence of Laguerre exponential polynomials converging in the $L_1$-sense to $g$. Indeed, by Lemma 1, we can construct, by the Gram–Schmidt orthonormalization process, an orthonormal sequence $(u_k)$ in $L_2(R_+)$ such that the sequence

$$p_n = \sum_1^n (h, u_k) \, u_k \, , \qquad n = 1, 2, \ldots,$$

of partial sums of the Fourier expansion of $h$ with respect to $(u_k)$ satisfies the conditions of Theorem 1.

## REFERENCE

1. I. C. GOHBERG AND M. G. KREIN, Systems of integral equations on a half-line with kernels depending on the difference of the arguments, *Amer. Math. Soc. Transl. Ser. 2* **14** (1960), 217–287.